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On General Conics

by

Sylvia E. Bowen and Nazir A. Warsi

SUMMARY: In this article, we have used the powerful method of tensor analysis to discuss conics in general and their properties.

1. INTRODUCTION

The general equation of the second degree in two variables represents a curve in two dimensions. The equation under certain conditions may represent a pair of straight lines, a parabola, an ellipse, a circle and a hyperbola. We shall start with the second degree equation and reduce it to the standard forms of equations of different conics. All the quantities are referred to the rectangular cartesian coordinate system. The Latin indices take values 1, 2 and a repeated index denotes summation unless otherwise mentioned. Also X^i denotes the coordinates of a point with reference to a standard point (origin). Hence, without any loss of generality, the general equation of the second degree can be written as:

(1.1) $g_{ij} x^i x^j + 2 f_i x^i + h = 0$

where g_{ij} is a covariant symmetric tensor or order two and f_i a covariant vector. Quantities g_{ij}, f_i and h do not contain X's.

Now, using the canonical transformations, we shall reduce (1.1) to the standard form.

2. CONDITIONS FOR DIFFERENT CONICS

Let us consider the pair of straight lines given by

(2.1) $g_{ij} x^i x^j = 0$

Let us also consider two points $P(x^i_1/)$ * and $Q(x^i_2/)$ which do not lie on the straight lines. If there is a point $R(x^i_3/)$ lying on the straight line PQ such that it divides it into two parts the ratio of which is λ, then obviously

(2.2) $$x^i_3/ = \frac{x^i_1/ + \lambda x^i_2/}{1+\lambda}$$

If $X^i_3/$ lies on the straight lines represented by (2.1), then, we have

(2.3) $$\lambda^2 g_{ij} x^i_2/ x^j_2/ + 2\lambda g_{ij} x^i_1/ x^j_2/ + g_{ij} x^i_1/ x^j_1/ = 0$$

which is a quadratic equation in λ. Hence, its two roots λ_1, and λ_2 satisfy the equations:

(2.4) $$\lambda_1 + \lambda_2 = \frac{-2 g_{ij} x^i_1/ x^j_2/}{g_{ij} x^i_2/ x^j_2/}$$

and

(2.5) $$\lambda_1 \lambda_2 = \frac{g_{ij} x^i_1/ x^j_1/}{g_{ij} x^i_2/ x^j_2/}$$

*Quantities under solidus denote different entities.

If the range is harmonic, then

$$(2.6)\quad \lambda_1 + \lambda_2 = 0$$

which in consequence of (2.4) gives

$$(2.7)\quad g_{ij}\, x_1^i /\, x_2^j / = 0$$

Clearly the locus of $x_2^j /$ is given by

$$(2.8)\quad g_{ij}\, x^i\, x_1^j / = 0$$

It is interesting to note that P and Q are independent of each other. Hence, these could be unit points in a new coordinate system. If we transform the two axes and denote new quantities by bars (—), we get

$$(2.9)\quad \overline{x}_1^i / = \delta_1^i\,,\quad \overline{x}_2^i / = \delta_2^i$$

where

$$(2.10)\quad \delta_i^j = \begin{Bmatrix} 1 \text{ when } i = j \\ 0 \text{ when } i \neq j \end{Bmatrix}$$

Also, the transformed form of (2.7) is

$$(2.11)\quad \overline{g_{ij}}\; \overline{x_1^i} /\; \overline{x_2^j} / = 0$$

which, inconsequence of (2.9), gives

$$(2.12)\quad \overline{g}_{12} = 0$$

The above type of transformation is known as Canonical Transformation.

REMARK (2.1): Since the points $P(x_1^i /)$ and $Q(x_2^i /)$ are arbitrary, we can have an infinite number of such transformations. With the help of transformations of these types, (1.1) can be written in the form

$$(2.13)\quad \overline{g}_{11}(\overline{x}^1)^2 + g_{22}(\overline{x}^2)^2 + 2\overline{f}_i\, \overline{x}^i + h = 0$$

Let us make a supposition that (1.1)* does not represent a pair of straight lines, that is:

$$(2.14)\quad \Delta = \begin{vmatrix} g_{11} & f_1 & g_{12} \\ f_1 & g_{22} & f_2 \\ g_{12} & f_2 & h \end{vmatrix} \neq 0$$

*The case of straight lines will be discussed in Section 3.

Hence, equation (2.13) will represent a circle, parabola, ellipse and hyperbola under different conditions.

If

$$(2.15) \qquad \bar{g}_{11} = \bar{g}_{22}$$

then (2.13) obviously represents a circle with center $\dfrac{-\bar{f}_1}{\bar{g}_{11}}, \dfrac{-\bar{f}_2}{\bar{g}_{11}}$

and radius $\sqrt{\dfrac{(\bar{f}_1)^2 + (\bar{f}_2)^2 - h(\bar{g}_{11})^2}{\bar{g}_{11}}}$

Now let $\bar{g}_{11} = 0$ and $\bar{g}_{22} \neq 0$. Hence, (2.13) can be put in the form:

$$(2.16) \qquad \left(\bar{x}^2 + \frac{\bar{f}_2}{\bar{g}_{22}}\right) = \frac{-2\bar{f}_1}{\bar{g}_{22}}\left\{\bar{x}^1 + \frac{h\bar{g}_{22} - (\bar{f}_2)^2}{2\bar{f}_1\,\bar{g}_{22}}\right\}$$

which represents a parabola with latus rectum $\dfrac{-2\bar{f}_1}{\bar{g}_{22}}$

and vertex $\left(\dfrac{-h\bar{g}_{22} - (\bar{f}_2)^2}{2\bar{f}_1\,\bar{g}_{22}}, \quad \dfrac{-\bar{f}_2}{\bar{g}_{22}}\right)$

If $\bar{g}_{22} = 0, \bar{g}_{11} \neq 0$, we can similarly show that the equation represents a parabola. It can be easily proved that conditions for a parabola are equivalent to $g = |g_{ij}| = 0$

Let $\bar{g}_{11} \neq 0$, $\bar{g}_{22} \neq 0$. Hence, (2.13) can be put in the form:

$$(2.17) \qquad \frac{\left(\bar{x}^1 + \dfrac{\bar{f}_1}{\bar{g}_{11}}\right)^2}{\left(\dfrac{\bar{f}_1}{\bar{g}_{11}}\right)^2 + \left(\dfrac{\bar{f}_2}{\bar{g}_{11}\bar{g}_{22}}\right) - \left(\dfrac{h}{\bar{g}_{11}}\right)} + \frac{\left(\bar{x}^2 + \dfrac{\bar{f}_2}{\bar{g}_{22}}\right)^2}{\left(\dfrac{\bar{f}_1}{\bar{g}_{11}\bar{g}_{22}}\right) + \left(\dfrac{\bar{f}_2}{\bar{g}_{22}}\right) - \left(\dfrac{h}{\bar{g}_{22}}\right)} = 1$$

which obviously represents an ellipse or a hyperbola according as:

a) denominators of both the terms on L.H.S. are positive

b) one is positive and the other negative

Also, both the denominators give the square of the principal axes of conics.

REMARKS (2.2): It can easily be shown that the above conditions for an ellipse or a hyperbola are equivalent to: $g > 0$ and $g < 0$

3. CENTRAL CONICS

We have seen in Section 2 that we can at once decide whether a given equation of the second degree represents the central conics (ellipse and hyperbola). In this section we shall discuss the details of these conics.

In case the given equation represents a central conic, let us assume that coordinates of the center are $x^i_c/$. With the help of the law of vector-addition we get

$$(3.1)\quad x^i = y^i + x^i_c/$$

where Y^i are coordinates with reference to a parallel system of axes but origin at the center.

In virtue of (1.1) and (3.1), we get:

$$(3.2)\quad g_{ij}\, y^i y^j + 2(g_{ij}\, x^j_c/ + f_i)\, y^i + g_{ij}\, x^i_c/\, x^j_c/ + 2 f_i\, x^i_c/ + h = 0$$

Putting the coefficient of the linear term equal to zero, we get a set of equations:

$$(3.3)\quad g_{ij}\, x^j_c/ + f_i = 0$$

which can be solved for $x^i_c/$. Let the cofactor of g_{ij} in g be G^{ji}. Multiplying (3.3) by G^{il} and summing with respect to i, we get:

$$(3.4)\text{a}\quad G^{il} g_{ij}\, x^j_c/ = -f_i\, G^{il}$$

or

$$(3.4)\text{b}\quad g\, \delta^l_j\, x^j_c/ = -f_i\, G^{il}$$

or

$$(3.4)\text{c}\quad g\, x^l_c/ = f_i\, G^{il}$$

which determines the center uniquely under the condition $g \neq 0$ * Obviously for $g = 0$ the center does not exist. This case will be discussed in the next section. In case the determinant g does not vanish, (3.4)c gives:

$$(3.5)\quad x^l_c/ = \frac{-f_i\, G^{il}}{g}$$

Now, from (3.2), (3.3) and (3.5) we have:

$$(3.6)\quad g_{ij}\, y^i y^j + \frac{1}{g}(gh - f_i f_l\, G^{li}) = 0$$

*This rule is known as Cramer's rule.

which represents a pair of straight lines when

$$(3.7)\quad gh - f_i f_l G^{li} = 0$$

It can be easily shown that (3.7) is equivalent to

$$(3.8)\quad \Delta = \begin{vmatrix} g_{11} & f_1 & g_{12} \\ f_1 & g_{22} & f_2 \\ g_{12} & f_2 & h \end{vmatrix} = 0$$

In case (3.7) is not true, (3.6) represents a central conic. Let us omit here the easy case of a circle. We shall use (3.6) to determine the ellipse and hyperbola. In both these cases we follow the same procedure.

With the help of canonical transformations discussed in section 2, we can put (3.6) in the form:

$$(3.9)\quad \bar{g}_{11}(\bar{q}^1)^2 + \bar{g}_{22}(\bar{q}^2)^2 + \frac{1}{g}\left(gh - f_i f_l G^{il}\right) = 0$$

The Squares of Axes of these conics are given by

$$(3.10)\quad A^2 = \frac{1}{g\,\bar{g}_{11}}\left(f_i f_l G^{li} - gh\right)$$

and

$$(3.11)\quad B^2 = \frac{1}{g\,\bar{g}_{22}}\left(f_i f_l G^{li} - gh\right)$$

A^2, B^2 both being positive, the smaller determines the minor, and the greater, the major axis of an ellipse. One being positive and the other negative, the former determines the transverse and the latter, the conjugate axis of a hyperbola.

4. THE PARABOLA

We have already seen in the preceding section that if G = O, center does not exist. This will be the case of a parabola. We shall mention here one of the very important properties of the parabola. The fact that the ratio of the square of the distance of a point on the curve from its axis to the distance of the same point from the tangent at the vertex gives the latus rectum, will be used subsequently for determination of the above-mentioned straight lines.

Let us put (1.1) in the aforesaid form. That is:

$$(4.1)\quad \left(\frac{q_i x^i + \lambda}{q}\right)^2 = \frac{\bar{q}}{(q)^2}\left(\frac{(p_i + 2\lambda q_i)\, x^i + \lambda^2 - h}{\bar{q}}\right)$$

where

$$(4.2)\quad ((g^{ij})) = ((q^i q^j))$$

$$(4.3)\quad (q_i)^2 = ((q_i q_i))$$

and

$$(4.4)\quad (\bar{q})^2 = (P_i + 2\lambda q_i)^2$$

Straight lines

$$(4.5)\quad q_i x^i + \lambda = 0$$

and

$$(4.6)\quad (P_i + 2\lambda q_i) x^i + \lambda^2 - h = 0$$

represent the axis and tangent at the vertex when they are at right angles; that is:

when

$$(4.7)\quad \lambda = \frac{-P_i q_i}{2(q)^2}$$

The latus rectum of the parabola is $\dfrac{\bar{q}}{(q)^2}$

It is obvious from the above discussions that the parabola is completely determined.

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