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On Geometry of Shock Waves in Lagrangian Coordinate System

by

Nazir A. Warsi

SUMMARY: In this paper, we have found the jump-conditions of flow parameters for the perfect gas flows in Lagrangian Coordinate System.

1. INTRODUCTION

Let us consider a shock surface S moving relative to a gas and enveloping a particle at a time t whose coordinates are given by

(1.1) $$h^i = h^i(t)$$

where motion is referred to the rectangular cartesian coordinate system.

A Latin index takes values 1, 2 and 3; whereas a Greek index takes 1 and 2. Throughout the discussion, a repeated index denotes summation unless mentioned otherwise.

Now, let $x^i = x^i(h^i, t)$ be the position of a fluid particle at a time t and let the position of the shock surface at the same time be given by $\bar{x}^i$, then we have

(1.2) $$\bar{x}^i = x^i(h^i, t)$$

The shock and the fluid velocities are given by

(1.3) $$\bar{u}^i = \frac{\partial \bar{x}^i}{\partial t}$$

and

(1.4) $$u^i = \frac{\partial x^i}{\partial t}$$

respectively. Now, differentiating (1.2) partially with respect to t, we get

(1.5) $$\bar{u}^i = \dot{h}^j \frac{\partial x^i}{\partial h^j} + u^i$$

In what follows, τ stands for the specific volume, p for the pressure, I for the specific enthalpy and X^i for the unit normal vector to the shock surface. Also, $f_1/$ and $f_2/$ denote quantities in front of and behind the shock surface.

2. JUMP-RELATIONS

In this connection, we have the following theorems:

THEOREM (2.1): *For the conservation of the mass at the shock surface, we have*

(2.1) $$\dot{h}_{n/}[\tau] + [U_{n/}] = 0$$

where $$U_{2n/} = U_{2/i}\, X^i$$

and $$[f] = f_{2/} - f_{1/}$$

PROOF: On substitution for $\frac{\partial x^i}{\partial h^l}$ from (1.2) b.[1], (1.5) gives

(2.2)a $$\bar{u}^i = \dot{h}^l \tau \delta^i_l + u^i$$

or

(2.2)b $$\bar{u}^i = \dot{h}^l \tau + u^i$$

Multiplying (2.2) by X^i and summing with respect to i , we get

(2.3) $$\bar{u}_{n/} = \dot{h}_{n/} \tau + U_{n/}$$

where

(2.4) $$F_{n/} = F_i X^i$$

Writing (2.2)b for the regions in front of and behind the shock surface, we get

(2.5)a $$\bar{U}_{n/} = \dot{h}_{n/} \tau_{2/} + U_{2n/}$$

or

(2.5)b $$V_{2n/} = U_{2n/} - \bar{U}_{n/}$$

where

(2.6) $$V_{2n/} = U_{2n/} - \bar{U}_{n/}$$

From (2.5), it is obvious that

(2.7) $$\dot{h}_{n/}(\tau_2 - \tau_{1/}) + (U_{2n/} - U_{1n/}) = 0$$

which is equivalent to (2.1).

Theorem (2.2): $-\dot{h}_{n/}$ *is the mass crossing the unit area of the shock surface normally from the front side to the back side.*

PROOF: It is obvious from (2.5).

THEOREM (2.3): *The law of conservation of momentum at the shock surface is given by*

$$(2.8)\text{a} \quad [p]\, x^i = -\dot{h}_{n/}\, [u^i]$$

or

$$(2.8)\text{b} \quad [p] = -\dot{h}_{n/}\, [u_{n/}]$$

or

$$(2.8)\text{c} \quad [p] = -\dot{h}_{n/}\, [\tau]$$

PROOF: Let us consider an elementary area dA of the shock surface. The mass crossing dA per unit time from the front side to the back side is $-\dot{h}_{n/}\, dA$. If $p_{\alpha/}$ be the fluid pressure in the region, then the force due to the pressure is $p_{\alpha/}\, dA$. Since the rate of change of momentum in time δt per unit area is equal to the impulse in that direction, therefore, we have

$$(2.9) \quad [p]\, x^i\, dA\, \delta t - \dot{h}_{n/}\, dA\, [u^i]\, \delta t = 0$$

which is equivalent to (2.8)a. Multiplying (2.8)a by x^i and summing with respect to i, we get (2.8)b which gives (2.8)c in virtue of (2.1).

THEOREM (2.4): *The components of velocity along the shock surface are continuous.*

PROOF: Multiplying (2.8)a by $x^i_{,\alpha}$ * and summing with respect to i, we get

$$(2.10)\text{a} \quad [u^i]\, x^i_{;\alpha} = 0$$

or

$$(2.10)\text{b} \quad [u_\alpha] = 0$$

which proves the theorem.

THEOREM (2.5): *The law of conservation of mass at the shock surface is also given by*

$$(2.11) \quad [u^i] + \dot{h}_{n/}\, [\tau]\, x^i = 0$$

---

*Space coordinates $\{x^i\}$ of the surface are taken as the functions of two parameters $\{y^\alpha\}$. Comma following the Greek indices denotes partial differentiation with respect to the surface parameters. $x^i_{,\alpha}$ will, therefore, be tangential to the shock surface.

PROOF: The velocity in the region $\beta/$ is given by

$$U_{\beta/}^{i} = U_{\beta n/} X^{i} + U_{\beta/a} x^{i}_{;a} \tag{2.12}$$

which, gives

$$[U^{i}] = [U_{n/}] X^{i} + [U_{a}] x^{i}_{;a} \tag{2.13}$$

In virtue of (2.10) and (2.1), (2.13) gives (2.11).

THEOREM (2.6): *The law of conservation of energy at the shock surface is given by*

$$[\tfrac{1}{2} V_{n/}^{2} + I] = 0 \tag{2.13a}$$

or

$$\tfrac{1}{2} \dot{h}_{n/} [\tau^{2}] + [I] = 0 \tag{2.13b}$$

I. *being the specific enthalpy.*

PROOF: Mathematical statement of the law is

$$-\dot{h}_{n/}\, dA\, [\tfrac{1}{2} V^{2} + I] = 0 \tag{2.14a}$$

or

$$[\tfrac{1}{2} V^{2} + I] = 0 \tag{2.14b}$$

From (2.1), (2.11) and (2.14)b, we get (2.13)a. Substituting the values of $[V_{n/}^{2}]$ from (2.5)b, we get (2.13)b.

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On Geometry of Gas-Flows in Lagrangian Coordinate System

by

Nazir A. Warsi

SUMMARY: In this paper, we have derived the conservation equations in Lagrangian coordinate system.

1. INTRODUCTION

In Lagrangian Coordinate System the position of a fluid particle in motion at any time is given by the rectangular cartesian coordinates x^i * which are functions of time and other parameters characterizing the initial position of the moving particle. Let us define a quantity h^i satisfying the equation

$$(1.1) \quad h^i = \int_{x^i(o,t)}^{x^i(h^i,t)} \rho \, dx^i$$

Differentiating (1.1) partially with respect to h^1, we get

$$(1.2)a \quad \delta^i_1 = \rho \frac{\partial x^i}{\partial h^1}$$

or

$$(1.2)b \quad \tau \delta^i_1 = \frac{\partial x^i}{\partial h^1}$$

where

$$(1.3) \quad \tau = \frac{1}{\rho}$$

In the above equations ρ is the density of the fluid at the point x^i. Also quantities h^1, h^2, h^3, t are independent and every other state variable is their function. In the following work P will stand for the pressure, u^i for the components of fluid velocity and η for the specific entropy of the fluid.

2. EQUATIONS OF CONTINUITY AND MOTION

In this connection we have the following theorems.

THEOREM (1.1): *For an unsteady fluid flows, we have*

*In this and what follows Latin indices take values 1, 2 and 3; and a repeated index denotes summation unless mentioned otherwise. Throughout the discussions the notations of C. E. Weatherburn used in Riemanian Geometry and Tensor Calculus have been followed.

(2.1)a $$\frac{\partial \rho}{\partial t} + \frac{\rho}{3} u^i_{,i} = 0$$

or

(2.1)b $$u^i \rho_{,i} + \frac{\rho}{3} u^i_{,i} = 0$$

or

(2.1)c $$\frac{\partial \tau}{\partial t} - \frac{\tau}{3} u^i_{,i} = 0$$

or

(2.1)d $$u^i \tau_{,i} - \frac{\tau}{3} u^i_{,i} = 0$$

PROOF: Differentiating (1.2)a partially with respect to t, we get

(2.2) $$\frac{\partial \rho}{\partial t} \frac{\partial x^i}{\partial h^l} + \rho \frac{\partial u^i}{\partial h^l} = 0$$

where

(2.3) $$u^i = \frac{\partial x^i}{\partial t}$$

Multiplying (2.2) by $\frac{\partial h^l}{\partial x^j}$, we get

(2.4)a $$\frac{\partial \rho}{\partial t} \delta^i_j + \rho \frac{\partial u^i}{\partial x^j} = 0$$

or

(2.4)b $$\frac{\partial \rho}{\partial t} + \frac{\rho}{3} u^i_{,i} = 0$$

from which (2.1)a and (2.1)b are obvious. Again putting $\frac{1}{\rho} = \tau$ we get (2.1)c which is trivially equivalent to (2.1)d.

LEMMA (2.1): *If F is a function of state variables and coordinates, then we have*

(2.5) $$\frac{\partial}{\partial t} \int_{v(t)} \rho F dv = \int_{v(t)} \rho \frac{\partial F}{\partial t} dv$$

where dv *is an elementary fluid volume at the point.*

PROOF: Left hand side of (2.5) is the total variation of the integrand within the volume V. This will be affected by the variations of two parts: (i) the variation of F with respect to t within V, and (ii) the variation of ρdv with respect to t. Contribution of the first part is $\int_{v(t)} \frac{\partial F}{\partial t} \rho dv$ and that of the second is zero, for ρdv denotes elementary fluid mass which is invariant with regard to time. Therefore, the total variation is $\int_{v(t)} \frac{\partial F}{\partial t} \rho dv$ and hence (2.5) is true. Now we have

THEOREM (2.2): *The law of conservation of momentum, for a viscous and non-isotropic fluid is given by*

$$\text{(2.6)a} \quad \rho \frac{\partial u_i}{\partial t} = \rho f_i - p_{,i} + E_{ij,j}$$

or

$$\text{(2.6)b} \quad \rho u_{i,j} u_j = \rho f_i - p_{,i} + E_{ij,j}$$

where f_i *is the external force per unit mass,* p *the pressure of the fluid,* u^i *the velocity vector and* E_{ij} *the viscosity tensor.*

PROOF: Putting u^i for F in (2.5), we get

$$\text{(2.7)} \quad \frac{\partial}{\partial t} \int_{v(t)} \rho u_i dv = \int \rho \frac{\partial u_i}{\partial t} dv$$

Again the law of conservation of momentum is:

$$\text{(2.8)} \quad \frac{\partial}{\partial t} \int_{v(t)} \rho u_i dv = \int_{v(t)} \rho f_i dv - \int_{s(t)} p x_i ds + \int_{s(t)} E_{ij} x_i ds$$

From (2.7) and (2.8) we get

$$\text{(2.9)a} \quad \int_{v(t)} \rho \frac{\partial u_i}{\partial t} = \int_{v(t)} \rho f_i dv - \int_{s(t)} p x_i ds + \int_{s(t)} E_{ij} x_i ds$$

or

$$\text{(2.9)b} \quad \int_{v(t)} \rho \frac{\partial u_i}{\partial t} = \int_{v(t)} \rho f_i dv - \int_{v(t)} p_{,i} dv + \int_{v(t)} E_{ij,j} dv$$

which, obviously, gives (2.6)a. The equation (2.6)a is equivalent to (2.6)b.

COROLLARY (2.1): *The equation of motion for a perfect fluid, with no external force, is given by*

$$\text{(2.10)a} \quad \rho \frac{\partial u_i}{\partial t} = -p_{,i}$$

or

$$\text{(2.10)b} \quad \rho u_{i,j} u^j = -p_{,i}$$

or

(2.10)c $u_{i,j}\, u^j + \tau\, p_{,i} = 0$

PROOF: Putting $E_{ij} = 0$ and $f_i = 0$ in (2.6), we get (2.10)a and (2.10)b. Again putting $\frac{1}{\rho} = \tau$, we get (2.10)c.

THEOREM (2.3): *If the coefficient of viscosity be given by* ν^{lm}_{ij} *, then the equations (2.6)a and (2.6)b can be put in the form:*

(2.11) $\rho \frac{\partial u_i}{\partial t} = \rho f_i - p_{,i} + \nu^{lm}_{ij,j} u_{l,m} + \nu^{lm}_{ij} u_{l,mj}$

PROOF: The viscosity tensor E_{ij} can be put in the form:

(2.12) $E_{ij} = \nu^{lm}_{ij}\, u_{l,m}$

Substituting the values of E_{ij} from (2.12) in (2.6)a we get (2.11).

THEOREM (2.4): *For homogeneous isotropic and nonisentropic flow, the equation of motion is given by*

(2.13) $\rho \frac{\partial u_i}{\partial t} = \rho f_i - p_{,i} + \frac{1}{3}\mu\, x^j{}_{,ji} + \mu\, u_{i,jj}$

being the coefficient of viscosity.

PROOF: For a homogeneous flow

(2.14) $\nu^{lm}_{ij,k} = 0$

and for an isotropic flow

(2.15) $\nu^{mn}_{rs} = \lambda\, \delta^{mn}\, \delta_{rs} + \mu \left(\delta^m_r \delta^n_s + \delta^n_r \delta^m_s \right)$

where each delta is unity for equal indices and zero for the different.

In the equation (2.15), λ and μ are the constants of elasticity. Again, for compressible flows, we have

(2.16) $\lambda + \frac{2}{3}\mu = 0$

In virtue of (2.14), (2.15) and (2.16), the equation (2.11) gives (2.13) which is an equation equivalent to Navier-Stokes equation.

THEOREM (2.5): *For an isentropic fluid flow, the equation of motion is given by*

(2.17) $\frac{\partial u_i}{\partial t} \left(\frac{\partial \eta}{\partial p} \right)_\rho = 2 u_i \left(\frac{\partial \eta}{\partial \rho} \right)_p (\log \rho)_{,i}$

PROOF: Assuming that the entropy $\eta = \eta(p, \rho)$ is momentarily constant along the stream line, we have

(2.18)a $$\frac{\partial \eta}{\partial t} = 0$$

or

(2.18)b $$u^i \eta_{,i} = 0$$

or

(2.18)c $$u^i \frac{p_{,i}}{\rho}\left(\frac{\partial \eta}{\partial p}\right)_\rho + u^i \left(\frac{\partial \eta}{\partial \rho}\right)(\log \rho)_{,i} = 0$$

Substituting the values of $\frac{p_{,i}}{\rho}$ from the equation of motion in (2.18)c, we get (2.17).

3. EQUATION OF ENERGY

We have the following theorems.

THEOREM (3.1): *For a viscous fluid flow, the equation of energy is given by*

(3.1) $$\rho \frac{\partial}{\partial t}\left(\frac{1}{2}u^2 + e\right) = -(p u^i)_{,i} + (u^i E_{ij})_{,j} + (K T_{,i})_{,i} + \rho f_i u^i$$

where K is the coefficient of thermal conduction and T is temperature.

PROOF: The law of conservation of energy can mathematically be given by

(3.2) $$\frac{\partial}{\partial t}\int_{v(t)} \rho\left(\frac{1}{2}u^2 + e\right)dv = -\int_{s(t)} p u^i x_i \, ds + \int_{s(t)} E_{ij} x^j u^i \, ds + \int_{v(t)} \rho f_i u^i \, dv + \int_{s(t)} K T_{,i} x^i \, ds$$

By means of the LEMMA (2.1), we have

(3.3) $$\int_{v(t)} \rho \frac{\partial}{\partial t}\left(\frac{1}{2}u^2 + e\right)dv = -\int_{s(t)} p u^i x_i \, ds + \int_{s(t)} E_{ij} x^j u^i \, ds + \int_{s(t)} K T_{,i} x^i \, ds + \int_{v(t)} \rho f_i u^i \, dv$$

Applying Green's Theorem, we get

(3.4) $$\int_{v(t)} \rho \frac{\partial}{\partial t}\left(\frac{1}{2}u^2 + e\right)dv = -\int_{v(t)} (p u^i)_{,i} \, dv + \int_{v(t)} (E_{ij} u^i)_{,j} \, dv + \int_{v(t)} (K T_{,i})_{,i} \, dv + \int_{v(t)} \rho f_i u^i \, dv$$

from which (3.1) follows at once.

THEOREM (3.2): *For the equation of energy we also have*

(3.5) $$\rho \frac{\partial}{\partial t}\left(\frac{1}{2}u^2 + e\right) = u^i\left(\nu^{lm}_{ij,j} + \nu^{lm}_{ij} w_{l,mj} + \rho f_i - p_{,i}\right) - p u^i_{,i} + \nu^{lm}_{ij} u^i_{,j} w_{l,m} + (K T_{,i})_{,i}$$

PROOF: From (2.12), substituting the values of the viscosity tensor in (3.1), we get (3.5).

THEOREM (3.3): *For a perfect fluid, with no external force, the law of conservation of energy is given by*

$$(3.6) \quad \rho \frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2 + e\right) = - u^i p_{,i} - p u^i_{,i}$$

PROOF: Putting $E_{i;j}, T, K$ and F_i equal to zero in (3.9), we at once get (3.6)

THEOREM (3.4): *For an isentropic flow the equation of energy is given by*

$$(3.7) \quad u^i \left(\frac{\partial \eta}{\partial \rho}\right)_p (\log)_{,i} = \left(\frac{\partial \eta}{\partial p}\right)_p \left(u \frac{\partial u}{\partial t} + \frac{\partial e}{\partial t} + \frac{p}{3} u^i_{,i}\right)$$

PROOF: Entropy is given by

$$(3.8) \quad \eta = \eta(p, \rho)$$

which is momentarily constant along the stream lines. Hence, we have

$$(3.9)\text{a} \quad u^i \left(\frac{\partial \eta}{\partial \rho}\right)_p p_{,i} + u^i \left(\frac{\partial \eta}{\partial \rho}\right)_p \rho_{,i} = 0$$

or

$$(3.9)\text{b} \quad u^i p_{,i} \left(\frac{\partial \eta}{\partial \rho}\right)_p + \frac{\partial \rho}{\partial t}\left(\frac{\partial \eta}{\partial \rho}\right)_p = 0$$

Substituting the value of $u^i p_{,i}$ from (3.6) and that of $\frac{\partial \rho}{\partial t}$ from (2.4)b, we get (3.7).

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