FACULTY RESEARCH EDITION

of

The Savannah State College Bulletin

Published by

The Savannah State College

Volume 20, No. 2 Savannah, Georgia December, 1966

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On Variation of Velocity and Pressure Behind and Along A Shock Surface In Lagrangian Coordinate System

by

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1. INTRODUCTION

The flow of a perfect gas in Lagrangian coordinate system is given by [1]

$$(1.1)\quad \dot{U}^{i}_{\alpha/}\,\tau_{\alpha/,i} - \frac{\tau_{\alpha/}}{3}\,\dot{U}^{i}_{\alpha/,i} = 0$$

$$(1.2)\quad \dot{U}^{i}_{\alpha/}\,U_{\alpha/i,j} + \tau_{\alpha/}\,p_{\alpha/,j} = 0$$

$$(1.3)\quad p_{\alpha/} = \exp\{\eta_{\alpha/}/(\,C_v)\}\,p^{\gamma}$$

$$(1.4)\text{a}\quad \partial\eta_{\alpha/}/\partial t = 0$$

$$(1.4)\text{b}\quad \eta_{\alpha/,i}\,\dot{U}^{i}_{\alpha/} = 0$$

where $f_{\alpha/}$ is a quantity in the region $\alpha/$ and 2/,1/ denote the regions behind and in front of the shock surface.

Also, the conservation equations across the shock surface are given by [2]

$$(1.5)\quad [\dot{U}^{i}] = -\dot{h}_{n/}\,\delta_{z/}\,\tau_{1/}\,X^{i}$$

$$(1.6)\quad [p] = -\dot{h}_{n/}\,\delta_{z/}\,\tau_{1/}$$

$$(1.7)\quad [\tau] = -\frac{\delta_{z/}}{\dot{h}_{n/}}\,V_{1n/}$$

where

$$(1.8)\quad \delta_{z/} = \frac{2}{\gamma+1}\cdot\frac{C^{2}_{1/} - \dot{h}^{2}_{n/}\,\tau_{1/}}{\dot{h}^{2}_{n/}\,\tau_{1/}}$$

2. VARIATIONS

Differentiating (1.3) and applying (1.4)b, we get

$$(2.1)\text{a}\quad \dot{U}^{i}_{\alpha/}\,p_{\alpha/,i} = C^{2}_{\alpha/}\,\rho_{\alpha/,i}\,\dot{U}^{i}_{\alpha/}$$

or

$$(2.2)\text{b}\quad \dot{U}^{i}_{\alpha/}\,p_{\alpha/,i} = -\frac{C^{2}_{\alpha/}}{\tau^{2}_{\alpha/}}\,\tau_{\alpha/,i}\,\dot{U}^{i}_{\alpha/}$$

In consequence of (1.1) and (2.1), we get

$$(2.3)\quad \dot{U}^{i}_{\alpha/}\,\dot{U}^{j}_{\alpha/}\,\dot{U}^{i}_{\alpha/,j} = \frac{C^{2}_{\alpha/}}{3}\,\dot{U}^{i}_{\alpha/,i}$$

Multiplying (1.2) by $\dot{U}^{j}_{\alpha/}$ and using (2.3), we get

$$(2.4)\quad \tau_{\alpha/}\,\dot{U}^{j}_{\alpha/}\,p_{\alpha/,j} = -\frac{C^{2}_{\alpha/}}{3}\,\dot{U}^{i}_{\alpha/,i}$$

Now, let us define quantities L^{i}_{j} such that

(2.5) $L^{i}_{\alpha} \stackrel{def}{=} x^{i}_{,\alpha}$

and

(2.6) $L^{i}_{3} \stackrel{def}{=} (u^{i}_{1/} - \dot{h}_{n/}\, \zeta_{1/}\, \delta_{z/} X^{i})$

If M^{i}_{j} are quantities such that $|L^{i}_{j}| \neq 0$, and

(2.7) $M^{i}_{k} L^{k}_{j} = \delta^{i}_{j}$

(2.8) $M^{k}_{i} M^{j}_{k} = \delta^{j}_{i}$,

then M^{i}_{j} are the inverse of L^{i}_{j}. Hence, we have

(2.9) $M^{i}_{j} = \dfrac{\text{cofactor of } L^{i}_{j} \text{ in } |L^{i}_{j}|}{|L^{i}_{j}|}$.

Now, from (2.5) and (2.6), we get

(2.10) $|L^{i}_{j}| = \begin{vmatrix} L^{1}_{1} & L^{1}_{2} & L^{1}_{3} \\ L^{2}_{1} & L^{2}_{2} & L^{2}_{3} \\ L^{3}_{1} & L^{3}_{2} & L^{3}_{3} \end{vmatrix}$

The values of L^{1}_{1}, L^{1}_{2}, etc. can be substituted from (2.5) and (2.6) and $|L^{i}_{j}|$ can be calculated. Let $L = |L^{i}_{j}|$.

We have the following theorems.

THEOREM 2. 1: *The values of* M^{α}_{i} *are given by*

(2.11)a $M^{\alpha}_{i} = \frac{1}{L} \epsilon^{\alpha\beta} \epsilon_{ijk} x^{j}_{,\beta} (u^{k}_{1/} - \dot{h}_{n/}\, \zeta_{1/}\, \delta_{z/} X^{k})$

or

(2.11)b $M^{\alpha}_{i} = \frac{1}{L} \epsilon^{\alpha\beta} \epsilon_{ijk} (u^{k}_{1/} + v_{1n/}\, \delta_{z/} X^{k})$

Where

$\epsilon^{\alpha\beta}$ *are quantities, having values*

(i) $+1$ when $\alpha = 1, \beta = 2$

(ii) -1 when $\alpha = 2, \beta = 1$

(iii) 0 when $\alpha = \beta$

and ϵ_{ijk} *are the components of an isotropic tensor having numerical values*

(i) $+1$ *when* ijk *is an even permutation of* 123

(ii) -1 *when* ijk is *an odd permutation of* 123

(iii) 0 *when* ijk *are the same.*

PROOF: From (2.9), we get

(2.12) $M^{\alpha}_{i} = \dfrac{\text{cofactor of } L^{i}_{\alpha} \text{ in } L}{L}$

In consequence of (2.5), (2.6) and (2.10) we get (2.11)a which by virtue of the relation $-\dot{h}_{n/}\,\tau_{1/} = V_{1n/}$, gives (2.11)b.

THEOREM 2.2: *Quantities* M^3_i *are given by*

$$(2.13)\quad M^3_i = \frac{1}{2L}\,\epsilon^{\alpha\beta} e_{\alpha\beta} X_i$$

where $e_{\alpha\beta}$ *are the components of a surface tensor skew-symmetric in* α, β *and defined by*

$$(2.14)\quad e_{\alpha\beta} \overset{\text{def}}{=} \epsilon_{ijk} X^i x^j_{,\alpha}\; x^k_{,\beta}$$

PROOF: From (2.9) we have

$$(2.15)\quad M^3_i = \frac{\text{Cofactor of } L^i_3 \text{ in } L}{L}$$

which with the help of (2.5), (2.6) and (2.10) gives (2.13).

THEOREM 2.3: *The variation of velocity along the surface behind the shock is given by*

$$(2.16)\quad u^i_{2/,j}\, x^j_{,\alpha} = u^i_{1/,j}\, x^j_{,\alpha} + \dot{h}_{n/}\tau_{1/}\delta_{\tau/} d_{\alpha\beta} g^{\beta\gamma} x^i_{,\gamma} + \frac{4C_{1/}C_{1/,\alpha}X^i}{(\gamma+1)\dot{h}_{n/}\tau_{1/}} + \left[\left\{4+\delta_{\tau/}(\gamma+1)\dot{h}^2_{n/}\tau^2_{1/} - 4C^2_{1/} + \dot{h}_{n/}\tau_{1/}\right\}\right] \frac{(\dot{h}_{n/}\tau_{1/})_{,\alpha}}{(\gamma+1)\dot{h}^2_{n/}\tau^2_{1/}},$$

where $g^{\alpha\beta}$ *is a reciprocal tensor of the first fundamental tensor* $g_{\alpha\beta}$ *of the surface and* $d_{\alpha\beta}$ *the second fundamental tensor of the surface.*

PROOF: Differentiating (1.5) and (1.8) partially with respect to the surface parameters y^α , we get

$$(2.17)\quad U^i_{2/,j}\, x^j_{,\alpha} = U^i_{1/,j}\, x^j_{,\alpha} - \dot{h}_{n/}\tau_{1/}\delta_{\tau/,\alpha}X^i - \dot{h}_{n/}\delta_{\tau/}\tau_{1/,\alpha}X^i - \dot{h}_{n/,\alpha}\delta_{\tau/}\tau_{1/}X^i - \dot{h}_{n}\tau_{1/}\delta_{\tau/}X^i_{,\alpha}$$

and

$$(2.18)\quad \delta_{\tau/,\alpha} = \frac{4}{(\gamma+1)\dot{h}^3_{n/}\tau^3_{1/}}\left\{\dot{h}_{n/}\tau_{1/}C_{1/}C_{1/,\alpha} + (\dot{h}_{n/}\tau_{1/})_{,\alpha}\left(\dot{h}^2_{n/}\tau^2_{1/} - C^2_{1/} - \tau_{1/}\dot{h}_{n/}\right)\right\}$$

By making use of Weingarten equation [3] in (2.17) and applying (2.18), we get (2.16).

THEOREM 2.4: *The variation of pressure along the surface behind the shock, is given*

$$(2.19)\quad p_{2/,j}\, x^j_{,\alpha} = p_{1/,\alpha} - (\dot{h}^2_{n/}\tau_{1/})_{,\alpha}\delta_{\tau/} - \dot{h}^2_{n/}\tau_{1/}\delta_{\tau/,\alpha}$$

PROOF. Differentiating (1.6) partially with respect to y^α, we get (2.19).

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On Vorticity Behind A Shock Surface In Lagrangian Coordinate System

by

Nazir A. Warsi

1. INTRODUCTION

The author [1] has discussed the variation of velocity and pressure along and behind the shock surface. The results derived by him will be used in this paper in finding the vorticity behind the shock wave.

2. DERIVATIVE OF VELOCITY

If S_{ij} are the quantities defined by

(2.1) $S_{ij} \overset{\text{def}}{=} u^{r}_{2/,s} L^{r}_{i} L^{s}_{j}$,

then we have the following theorems.

Theorem 2.1: *Quantities* $S_{\alpha\beta}$ *are given by*

(2.2) $S_{\alpha\beta} = u_{1/r,s} x^{r}_{,\alpha} x^{s}_{,\beta} + \dot{h}_{n/} \tau_{1/} \delta_{\tau/} d_{\alpha\beta}$

Proof: Putting $i = \alpha$, $j = \beta$ in (2.1), we get

(2.3) $S_{\alpha\beta} = u^{r}_{2/,s} L^{r}_{\alpha} L^{s}_{\beta}$

which, in consequence of (2.5) of [1] gives

(2.4) $S_{\alpha\beta} = u^{r}_{2/,s} x^{r}_{,\alpha} x^{s}_{,\beta}$

On substituting from (2.16) of [1] and using $X^{i} x^{i}_{,\alpha} = 0$,

(2.4) gives (2.2).

Theorem 2.2: *Quantities* $S_{\alpha 3}$ *are given by*

(2.5) $S_{\alpha 3} = \tau_{1/} (1 + \delta_{\tau/}) \{ p_{1/,\alpha} - (\dot{h}^{2}_{n/} \tau_{1/})_{,\alpha} \delta_{\tau/} - \dot{h}^{2}_{n/} \tau_{1/}$

Proof: Putting $i = \alpha$, $j = 3$ in (2.1), we get

(2.6) $S_{\alpha 3} = u^{r}_{2/,s} L^{r}_{\alpha} L^{s}_{3}$

which, with the help of (2.5) and (2.6) of [1] gives

(2.7) $S_{\alpha 3} = u^{r}_{2/,s} x^{r}_{,\alpha} (u^{s}_{1/} - \dot{h}_{n/} \tau_{1/} \delta_{\tau/} X^{s})$.

We get (2.5) if we substitute from (2.16) of [1] in (2.7).

Theorem 2.3: *The quantities* $S_{3\alpha}$ *are given by*

$$(2.8)\quad S_{3\alpha} = (u^{r}_{1/} - \dot{h}_{n/} \tau_{1/} \delta_{\tau/} X^{r}) \Big\{ u_{1/,s} x^{s}_{,\alpha} + \dot{h}_{n/} \tau_{1/} \delta_{\tau/} d_{\alpha\beta} g^{\beta r} x^{r}_{,r} + \frac{4 c_{1/} c_{1/,\alpha} X^{r}}{(\gamma+1) \dot{h}_{n/} \tau_{1/}} + \frac{(\dot{h}_{n/} \tau_{1/})_{,\alpha}}{(\gamma+1) \dot{h}_{n/} \tau_{1/}} \cdot \Big[\{ 4 + \delta_{\tau/} (\gamma+1) \dot{h}^{2}_{n/} \tau^{2}_{1/} - 4 c^{2}_{1} + \dot{h}_{n/} \tau_{1/} \} \Big] $$

Proof: Putting $i = 3$, $j = \alpha$ in (2.1), we get

(2.9) $S_{3\alpha} = U^{r}_{2/,s} L^{r}_{3} L^{s}_{\alpha}$,

which, by virtue of (2.5) and (2.6) of [1], gives

(2.10) $S_{3\alpha} = U^{r}_{2/,s} X^{r}_{,\alpha} (U^{s}_{1/} - \dot{h}_{n/} \bar{\tau}_{1/} \delta_{\tau/} X^{s})$.

using (2.17) of [1], we get (2.8).

Theorem 2.4: *The quantity* S_{33} *is given by*

(2.11) $S_{33} = \left\{ C^{2}_{1/} - \frac{\gamma-1}{2} \delta_{\tau/} (\delta_{\tau/} + 2) \dot{h}^{2}_{n/} \bar{\tau}^{2}_{1/} \right\} \frac{U^{i}_{2/,i}}{3}$

Proof: Putting $i = 3$, $j = 3$ in (2.1), we get

(2.12) $S_{33} = U^{r}_{2/,s} L^{r}_{3} L^{s}_{3}$

which, in consequence of (2.6) of [1], gives

(2.13) $S_{33} = U^{r}_{2/,s} (U^{r}_{1/} - \dot{h}_{n/} \bar{\tau}_{1/} \delta_{\tau/} X^{r})(U^{s}_{1/} - \dot{h}_{n/} \bar{\tau}_{1/} \delta_{\tau/} X^{s})$

On substitution from (2.17) of [1], (2.13) gives (2.11).

Theorem 2.5: *The value of* $U^{i}_{2/,i}$ *is given by*

(2.14) $U^{i}_{2/,i} = \dfrac{3L^{2}\{S_{\alpha\beta} M^{\alpha}_{i} \cdot M^{\beta}_{i} + 2 S_{(\alpha 3)} M^{\alpha}_{i} M^{3}_{i}\}}{3L^{2} - 4e^{2}_{/2} \{C^{2}_{1/} - \frac{\gamma-1}{2} \delta_{\tau/} (\delta_{\tau/}+2) \dot{h}^{2}_{n/} \bar{\tau}^{2}_{1/}}$

Proof: From (2.1), (2.7) and (2.8), we get

(2.15) $U^{r}_{2/,s} = S_{ij} M^{i}_{r} M^{j}_{s}$

which, on putting s=r, yields

(2.16)a $U^{r}_{2/,r} = S_{ij} M^{i}_{r} M^{j}_{r}$

or

(2.16)b $U^{r}_{2/,r} = S_{\alpha\beta} M^{\alpha}_{r} M^{\beta}_{r} + 2 S_{(\alpha 3)} M^{\alpha}_{r} M^{\beta}_{r} + S_{33} M^{3}_{r} M^{3}_{r}$.

By virtue of (2.13) of [1], (2.12) and (2.16)b, we get (2.14).

Theorem 2.6: *The quantity* S_{33} *is completely determined in terms of the quantities of region, in front of the shock surface, with the help of* (2.17)

$$S_{33} = \frac{3L^{2}\{C^{2}_{1/} - \frac{\gamma-1}{2} \delta_{\tau/} (\delta_{\tau/}+2) \dot{h}^{2}_{n/} \bar{\tau}^{2}_{1/}\}\{S_{\alpha\beta} M^{\alpha}_{i} M^{\beta}_{i} + 2 S_{(\alpha 3)} M^{\alpha}_{i} M^{3}_{i}\}}{3L^{2} - 4e^{2}_{/2}\{C^{2}_{1/} - \frac{\gamma-1}{2} \delta_{\tau/} (\delta_{\tau/}+2) \dot{h}^{2}_{n/} \bar{\tau}^{2}_{1/}\}}$$

Proof: Applying (2.14) to (2.11), we get (2.17).

Theorem 2.7: *The components of vorticity vector* $\omega^{k}_{2/}$ *behind the shock surface are given by*

(2.18) $\omega^{k}_{2/} \in^{ijk} S_{lm} M^{l}_{i} M^{m}_{j}$

Proof: From all the discussions of article 2, it is obvious that M_{j}, S_{ij} are completely determined in terms of the quantities of the region in front of the shock surface. Hence from (2.1) and (2.7), (2.8) of [1], we get

(2.19) $U^{i}_{2/,j} = S_{lm} M^{l}_{i} M^{m}_{j}$

The vorticity vector $\omega^i_{\alpha/}$ is given by

(2.20) $\omega^i_{\alpha/} = \epsilon^{ijk} u^k_{\alpha/,j}$

Substituting from (2.19) for α =2 in (2.20), we get (2.19).

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On Gradients of Specific Volume and Pressure Behind A Shock Surface In Lagrangian Coordinate System

by

Nazir A. Warsi

1. INTRODUCTION

The author [1, 2] has discussed the variation of flow parameters and vorticity behind a shock surface in Lagrangian Coordinate System. This article deals with the gradients of pressure and density using the results obtained in [1, 2].

2. GRADIENT OF DENSITY

The specific volume shock strength is given by [3]

(2.1) $\delta_{\tau/} = [\tau]/\tau_{1/}$

Differentiating (2.1) partially with respect to y^α, we get

(2.2) $\tau_{2/,j}\, x^j_{,\alpha} = \tau_{1/,\alpha}(1+\delta_{\tau/}) + \tau_{1/}\,\delta_{\tau/,\alpha}.$

Also, we have [4, 3]

(2.3) $u^i_{2/}\,\tau_{2/,i} - \frac{\tau_{2/}}{3}\, u^i_{2/,i} = 0$

(2.4) $[u^i] = -h_{n/}\,\delta_{\tau/}\,\tau_{1/}\, X^i$

On substitution from (2.3), (2.1) and (2.4), the equation (2.2) becomes (2.5) $\tau_{2/,j}\,(u^j_{1/} - h_{n/}\,\tau_{1/}\,\delta_{\tau/}\, X^j) = \tau_{1/}(1+\delta_{\tau/})\, u^i_{2/,i}$

which, in consequence of (2.14) of [2], becomes

(2.6) $\tau_{2/,j}\,(u^j_{1/} - h_{n/}\,\tau_{1/}\,\delta_{\tau/}\, X^j) = \frac{\tau_{1/}(1+\delta_{\tau/})\, 3L^2}{3L^2 - 4e^2_{12}} \frac{\{\delta_{\alpha\beta} M^\alpha_i M^\beta_i + 2\delta_{(\alpha 3)} M^\alpha_i M^3_i\}}{\{c^2_{1/} - \frac{\gamma-1}{2}\,\delta_{\tau/}(\delta_{\tau/}+2)\, h^2_{n/}\,\tau^2_{1/}\}}$

Now, let us define quantities Z_i such that

(2.8) $Z_i = \tau_{2/,j}\, L^j_i$

Hence, we have the following theorem.

Theorem 2.1: *The quantities* Z_α *are given by the equation*

(2.9) $Z_\alpha = \tau_{1/,\alpha}(1+\delta_{\tau/}) + \tau_{1/}\,\delta_{\tau/,\alpha}$

Proof: Putting $i=\alpha$ in (2.8), we get

(2.10) $Z_\alpha = \tau_{2/,j}\, L^j_\alpha$

which, with the help of (2.5) of [1], becomes

(2.11) $Z_\alpha = \tau_{2/,j}\, x^j_{,\alpha}$

Equation (2.9) is readily obtained if we substitute from (2.6) in (2.11).

Theorem: 2.2: *The quantity* $\mathcal{Z}_3$ *is given by*

(2.12) $$\mathcal{Z}_3 = \frac{\tau_{1/}(1+\delta\tau_{1/})3L^2\{S_{\alpha\beta} M^{\alpha}_{i} M^{\beta}_{i} + 2S_{(\alpha 3)} M^{\alpha}_{i} M^{3}_{i}\}}{3L^2 - 4e_{/2}^{2}\{c_{1/}^{2} - \frac{\gamma-1}{2}\delta\tau_{/}(\delta\tau_{/}+2)h^{2}_{n/}\tau^{2}_{1/}\}}$$

Proof: Putting $i = 3$ in (2.8), we get

(2.13) $$\mathcal{Z}_3 = \tau_{2/,j} L^{j}_{3},$$

which, in consequence of (2.6) of [1], becomes

(2.14) $$\tau_{2/,j}(u^{j}_{1/} - h_{n/}\tau_{1/}\delta\tau_{/}X^{j})$$

By virtue of (2.14) and (2.6), we get (2.12).

Theorem 2.3: *The gradient of the specific volume* τ *is given by*

(2.15) $$\tau_{2/,j} = \mathcal{Z}_{i} M^{i}_{j}$$

Proof: From the discussions of section 2, it is obvious that the quantities $\mathcal{Z}_i$ are completely known. From (2.8) and (2.7) of [1], we, at once, get (2.15).

3. GRADIENT OF PRESSURE

We have the following theorems.

Theorem 3.1: *The gradient of pressure behind the shock surface is given by*

(3.1) $$p_{2/,i} = (u^{j}_{1/} - h_{n/}\tau_{1/}\delta\tau_{/}X^{j})S_{lm} M^{l}_{i} M^{m}_{j}.$$

Proof: The equation of motion behind the shock is given by [4]

(3.2) $$u^{i}_{2/} u_{2/i,j} + \tau_{2/} p_{2/,j} = 0$$

which, in turn, gives

(3.3) $$p_{2/,i} = \frac{-u^{j}_{2/} u_{2/j,i}}{\tau_{2/}}$$

In consequence of (2.1), (2.4) and (2.19) of [2], the equation (3.3) easily gives (3.1).

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