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ON CURVED SHOCK WAVES IN 3-DIMENSIONAL UNSTEADY FLOW OF CONDUCTING GASES

by

Nazir A. Warsi

ABSTRACT:-In this paper we have determined the first and the second partial derivatives of the density, the pressure, the velocity and the magnetic field vector. Consequently, the current density, the vorticity and the curvature and the torsion of the stream-lines and the magnetic lines of force behind the shock have been calculated.

1.INTODUCTION:-A shock surface divides the flow in two regions.If the flow in front of the shock is known,the flow immediately behind it can be determined with the help of the usual conservation laws and the transport equations.If the coordinate of a point be x^i, then on the shock surface $x^i = x^i(y^\alpha)$†,where y^α are the surface parameters.With the help of usual tensor notations,conservation laws and transport equations,in the case of unsteady flow of conducting gases,can be written as [1,2]

(1) $\frac{\partial H_i}{\partial t} + u_j H_{i,j} - H_j u_{i,j} + H_i u_{k,k} = 0$

(2) $H_{i,i} = 0$

(3) $\frac{\partial \rho}{\partial t} + u_i \rho_{,i} + \rho u_{i,i} = 0$

(4) $\rho \frac{\partial u_i}{\partial t} + \rho u_j u_{i,j} + p^*_{,i} - \frac{1}{4\pi} H_j H_{i,j} = 0$

(5) $\frac{\partial \eta}{\partial t} + u^i \eta_{,i} = 0$

and the equation of the state of gases with constant specific heats as [3]

(6) $p = \exp(\eta / J c_v)\rho^\gamma$

In the above equations H_i stands for the components of the magnetic field, u_i for the velocity components, p^* for the expression $p + H^2/(8\pi)$, p for the pressure, ρ for the density, η for the entropy of the fluid and γ for the ratio of the specific heats at constant pressure and volume.A comma following a Latin index denotes the partial differentiation with respect to the Cartesian Co-ordinate.

In the following discussions if f and f_1 denote quantities behind and infront of the shock respectively,then the jump from the front side to the back side is given by

(7) $[f] = f - f_1$

Let X^i be the unit normal vector to the surface and $x^i_{,\alpha}$†† the projection vector to the tangent plane.Also,let us put

$$(8)\quad \begin{aligned} &a)\ u_n = u_i X_i,\quad u_{1n} = u_{1i} X_i \\ &b)\ H_n = H_i X_i,\quad H_{1n} = H_{1i} X_i \\ &c)\ u_\alpha = u_i x^i_{,\alpha},\quad H_\alpha = H_i x^i_{,\alpha} \\ &d)\ u^\alpha = g^{\alpha\beta} u_\beta,\quad H^\alpha = g^{\alpha\beta} H_\beta \end{aligned}$$

†In this and in what follows Latin indices take values 1,2,3 whereas Greek 1,2.For both a repeated index denotes summation.

††A comma following a Greek index denotes partial differentiation with respect to the surface parameters.

where $g^{\alpha\beta}$ is the reciprocal of the first fundamental tensor of the surface.

The shock conditions, in the case of 3-dimensional unsteady flow of conducting gases are given by [2,4]

$$(9) \quad \rho_1 v_{1n} = \rho v_n = m$$

$$(10) \quad H_{1n} = H_n$$

$$(11) \quad \rho_1 v_{1n} [v_i] = -[p^*] X_i + \frac{1}{4\pi} [H_i]$$

$$(12) \quad H_{1n} [v_i] = [H_i v_n]$$

$$(13) \quad \rho_1 v_{1n} \left[\frac{v^2}{2} + h + \frac{H^2}{4\pi\rho} \right] - [H_i v_i] H_{1n} = 0$$

In the above shock relations h and v_i are the specific enthalpy and the relative fluid velocity with respect to the shock surface respectively. If C_i be the shock velocity vector, then we have

$$(14) \quad v_i = u_i - c_i$$

2.FLOW PARAMETERS, VELOCITY AND MAGNETIC FIELD VECTORS.

Let us define quantities δ by

$$(15) \quad [\rho] = \delta \rho_1$$

By virtue of (15), (9) gives

$$(16) \quad [u_n] = \frac{-\delta}{1+\delta} v_{1n}$$

If we multiply (11) by X_i and make use of (10), we get

$$(17) \quad [p^*] = \frac{m\delta}{1+\delta} v_{1n}$$

In consequence of (11), (12) and (15), we get

$$(18) \quad [H_i] = \frac{4\pi m \delta H_{1n} v_{1n} X_i - 2\{H_{1n}^2(1+\delta) - 2\pi m \delta v_{1n}\} H_i}{H_{1n}^2(1+\delta) - 4\pi m v_{1n}}$$

and

$$(19) \quad [u_i] = \frac{(4\pi m v_{1n} - \delta H_{1n}^2) X_i}{\{H_{1n}^2(1+\delta) - 4\pi m v_{1n}\}(1+\delta)} - \frac{H_{1n}}{2\pi} \frac{\{H_{1n}^2(1+\delta) - 2\pi m \delta v_{1n}\}}{H_{1n}^2(1+\delta) - 4\pi m v_{1n}} H_{1i}$$

The value of specific enthalpy is equal to $\frac{\gamma}{\gamma-1} p/\rho$. If we substitute values of quantities of the region behind the shock, from the above equations in (13), an equation in δ is obtained. Thus it is obvious that the velocity, the pressure, the density and the magnetic field strength behind the shock are completely known if the flow ahead of the shock is known.

3. DIFFERENTIATION OF SHOCK RELATIONS.

Differentiating (15),(17),(18) and (19) with respect to y^α we get

(20) $\rho_{,i} x^i_{,\alpha} = \rho_{1,i} x^i_{,\alpha} + (\delta \rho_1)_{,\alpha} = A_\alpha$

(21) $p^*_{,i} x^i_{,\alpha} = p_{1,i} x^i_{,\alpha} + \left(\frac{m\delta}{1+\delta} v_{1n_1}\right)_{,\alpha} = B_\alpha$

(22) $H_{i,j} x^j_{,\alpha} = H_{1\,i,j} x^j_{,\alpha} + \left\{ \frac{4\pi m H_{1n_1} v_{1n_1} X_i - 2(H^2_{1n_1}\overline{1+\delta} - 2\pi m \delta v_{1n_1})}{H^2_{1n_1}(1+\delta) - 4\pi m v_{1n_1}} \right\}_{,\alpha} = C_{i\alpha}$

(23) $u_{i,j} x^j_{,\alpha} = u_{1\,i,j} x^j_{,\alpha} + \left\{ \frac{(4\pi m v_{1n_1} - \delta H^2_{1n_1} m \delta v_{1n_1}) m \delta v_{1n_1} X_i}{[H^2_{1n_1}(1+\delta) - 4\pi m v_{1n_1}](1+\delta)} - \frac{H_{1n_1}}{2\pi} \frac{[H^2_{1n_1}(1+\delta) - 2\pi m \delta v_{1n_1}] H_1}{H^2_{1n_1}(1+\delta) - 4\pi m v_{1n_1}} \right.$

In the above equations Weingarten Formula $X_{,\alpha} = -d_{\alpha\beta} g^{\beta\gamma} x^i_{,\gamma}$ ($d_{\alpha\beta}$ being the second fundamental tensor of the surface) has been used.

Before proceeding to determine derivatives, we shall put the equations of motion in proper forms. In this process the following additional results will be required.

(24) $u_i = u_{n_1} X_i + u^\alpha x^i_{,\alpha}$

(25) $H_i = H_{n_1} X_i + H^\alpha x^i_{,\alpha}$

and

(26) $\frac{\delta f}{\delta t} = \frac{\partial f}{\partial t} + c X_i \frac{\partial f}{\partial x^i}$

In (26) C is the magnitude of the shock velocity. In consequence of (24), (25), (26), (22) and (23), (1) gives

(27) $(u_{n_1} - c) H_{i,j} X_j - H_{n_1} u_{i,j} X_j + u^\beta C_{i\beta} - H^\beta D_{i\beta} + H_i u_{k,k} + \frac{\delta H_i}{\delta t} = 0$

Multiplying (4) by $x^i_{,\alpha}$, substituting for $p^*_{,i} x^i_{,\alpha}$ from (21) and applying (26), (24) and (25), we get

(28) $\rho(u_{n_1} - c) x^i_{,\alpha} X_j u_{i,j} - \frac{H_{n_1}}{4\pi} H_{i,j} x^i_{,\alpha} X_j + \rho u^\beta D_{i\beta} x^i_{,\alpha} - \frac{H_{n_1} H^\beta}{4\pi} C_{i\beta} x^i_{,\alpha} + B_\alpha + \rho \frac{\delta u_i}{\delta t} x^i_{,\alpha} =$

Multiplying (27) by X_i and taking help of (28), we get

(29) $u_{i,j} x^i_{,\alpha} X_j = E_\alpha + E'_\alpha u_{k,k} + E''_{i\alpha} \frac{\delta H_i}{\delta t} + E'''_{i\alpha} \frac{\delta u_i}{\delta t}$

and

(30) $H_{i,j} x^i_{,\alpha} X_j = F_\alpha + F'_\alpha u_{k,k} + F''_{i\alpha} \frac{\delta H_i}{\delta t} + F'''_{i\alpha} \frac{\delta u_i}{\delta t}$

where,

(31) $E_\alpha = \frac{H_{n_1} \{ u^\beta + (c - u_{n_1}) H^\beta \} C_{i\beta} x^i_{,\alpha} + \{ 4\pi\rho(u_{n_1} - c) u^\beta - H_{n_1} H^\beta \} D_{i\beta} x^i_{,\alpha} + 4\pi B_\alpha}{H^2_{n_1} - 4\pi\rho(u_{n_1} - c)^2}$

(32) $E'_\alpha = \frac{H_{n_1} H_\alpha}{H^2_{n_1} - 4\pi\rho(u_{n_1} - c)^2}$

$$(33)\; E''_{i\alpha} = \frac{H_{n/}\, x^i_{,\alpha}}{H^2_{n/} - 4\pi\rho(u_{n/}-c)^2}$$

$$(34)\; E'''_{i\alpha} = \frac{4\pi\rho(u_{n/}-c)\,x^i_{,\alpha}}{H^2_{n/} - 4\pi\rho(u_{n/}-c)^2}$$

$$(35)\; F_\alpha = \frac{\{4\pi\rho(u_{n/}-c) - 4\pi H^2_{n/} H_\beta\}\, C_{i\beta}\, x^i_{,\alpha} + 4\pi\{\rho H_{n/} u^\beta - \rho(u_{n/}-c)H^\beta\}\, D_\beta\, x^i_{,\alpha} + 4\pi H_{n/}\, B_\alpha}{H^2_{n/} - 4\pi\rho(u_{n/}-c)^2}$$

$$(36)\; F_\alpha = \frac{4\pi\rho(u_{n/}-c)}{H^2_{n/} - 4\pi\rho(u_{n/}-c)}$$

$$(37)\; F''_{i\alpha} = \frac{4\pi\rho(u_{n/}-c)\,x^i_{,\alpha}}{H^2_{n/} - 4\pi\rho(u_{n/}-c)^2}$$

$$(38)\; F'''_{i\alpha} = \frac{4\pi\rho\, H_{n/}\, x^i_{,\alpha}}{H^2_{n/} - 4\pi\rho(u_{n/}-c)^2}$$

Differentiating (6) materially and applying (5),(3) and (26), we get

$$(39)\; (cX_i - u_i)\, p_{,i} = \frac{\partial p}{\partial t} + \gamma\, p\, u_{k,k}$$

Applying (28) to (4), multiplying by $cX_i - u_i$ and substituting from (39), we get an equation which in consequence of (24),(25),(22),(23), (29) and (30), gives

$$(40)\; u_{i,j} X_i X_j = G + G' u_{k,k} + G''_i \frac{\partial H_i}{\partial t} + G'''_i \frac{\partial u_i}{\partial t} + \frac{1}{\rho(c-u_{n/})}\frac{\partial p}{\partial t}$$

where,

$$(41)\; G = \Big[4\pi\rho(c-u_{n/})u^\alpha E_\alpha + \{(c-u_{n/})H^\alpha - H_{n/}u^\alpha\}F_\alpha + 4\pi\rho(cX_i - u_i)u^\alpha D_{i\alpha} + C_{i\alpha}\{(cX_i - u_i)H^\alpha - H_i u^\alpha\}\Big] \Big/ 4\pi\rho(c-u_{n/})$$

$$(42)\; G' = \frac{4\pi\rho E'_\alpha (c-u_{n/})u^\alpha + \{(c-u_{n/})H^\alpha - H_{n/}u^\alpha\}F'_\alpha + 4\pi\gamma p}{4\pi\rho(c-u_{n/})}$$

$$(43)\; G''_i = \frac{4\pi\rho(c-u_{n/})E''_{i\alpha}u^\alpha + \{(c-u_{n/})H^\alpha - H_{n/}u^\alpha\}F''_{i\alpha}}{4\pi\rho(c-u_{n/})}$$

$$(44)\; G'''_i = \frac{4\pi\rho(c-u_{n/})E'''_{i\alpha}u^\alpha + 4\pi\rho(cX_i - u_i) + \{(c-u_{n/})H^\alpha - H_{n/}u^\alpha\}F''_{i\alpha}}{4\pi\rho(c-u_{n/})}$$

Again, if we multiply (27) by X_i and apply (40), we get

$$(45)\quad H_{i,j}X_iX_j=(u^{\beta}C_{i\beta}X_i-H^{\beta}D_{i\beta}X_i-GH_{n/})+u_{k,k}H_{n/}(1+G')+(X_i-G''_iH_{n/})\frac{\delta H_i}{\delta t}-H_{n/}G'''_i\frac{\delta u_i}{\delta t}-\frac{H_{n/}}{\rho(c-u_{n/})}\frac{\delta p}{\delta t}$$

4. PARTIAL DERIVATIVES OF U_i AND H_i

Let us define quantities L^i_{j} by

$$(46)\quad L^i_{\alpha}=x^i_{,\alpha}\quad \text{and}\quad L^i_3=X^i=X_i$$

If $|L^i_j|=L\neq 0$, then we can define quantities M^i_j by

$$(47)\quad M^i_kL^k_j=\delta^i_j\quad \text{and}\quad M^k_jL^i_k=\delta^i_j$$

Hence, quantities M^i_j are given by

$$(48)\quad M^{\alpha}_i=\frac{1}{L}\epsilon^{\alpha\beta}\epsilon_{ijk}x^j_{,\beta}X^k\quad \text{and}\quad M^3_i=\frac{1}{2L}\epsilon^{\alpha\beta}\epsilon_{\alpha\beta}X_i$$

where, $\epsilon_{\alpha\beta}$ stands for $\epsilon_{ijk}X^ix^j_{,\alpha}x^k_{,\beta}$. In these expressions, ϵ_{ijk} is having values

1) $+1$ when i,j,k form even permutations of 1,2,3
2) -1 when i,j,k form odd permutations of 1,2,3
3) 0 when i,j,k are not different

and $\epsilon^{\alpha\beta}$

1) $+1$ when $\alpha=1,\ \beta=2$
2) -1 when $\alpha=2,\ \beta=1$
3) 0 when $\alpha\neq\beta$

Again, let us define quantities K_{ij} and N_{ij} by

$$(49)\quad K_{ij}=H_{l,m}L^l_iL^m_j$$

$$(50)\quad N_{ij}=u_{l,m}L^l_iL^m_j$$

The above two equations, with the help of (46),(22),(23),(29),(30), (40) and (45), give

$$(51)\quad K_{\alpha\beta}=C_{i\beta}x^i_{,\alpha}$$

$$(52)\quad K_{\alpha 3}=F_{\alpha}+F'_{\alpha}u_{k,k}+F''_{i\alpha}\frac{\delta H_i}{\delta t}+F'''_{i\alpha}\frac{\delta u_i}{\delta t}$$

$$(53)\quad K_{3\alpha}=C_{i\alpha}X_i$$

$$(54)\quad K_{33}=u^{\beta}C_{i\beta}X_i-H^{\beta}D_{i\beta}X_i-GH_{n/}+H_{n/}(1+G')u_{k,k}+(X_i-G''_iH_{n/})\frac{\delta H_i}{\delta t}-H_{n/}G'''_i\frac{\delta u_i}{\delta t}-\frac{H_{n/}}{\rho(c-u_{n/})}\frac{\delta p}{\delta t}$$

$$(55)\quad N_{\alpha\beta}=D_{i\beta}x^i_{,\alpha}$$

$$(56)\quad N_{\alpha 3} = E_\alpha + E'_\alpha u_{k,k} + E''_{i\alpha} \frac{\delta H_i}{\delta t} + E'''_{i\alpha} \frac{\delta u_i}{\delta t}$$

$$(57)\quad N_{3\alpha} = D_{i\alpha} X_i$$

$$(58)\quad N_{33} = G + G' u_{k,k} + G''_i \frac{\delta H_i}{\delta t} + G''' \frac{\delta u_i}{\delta t} + \frac{1}{\rho(c-u_{n_1})} \frac{\delta p}{\delta t}$$

With the help pf the equations (51),(52),(53),(54),(55),(56), (57) and (58), quantities K_{ij} and N_{ij} are known in terms of the flow parameters behind the shock, the operator $\delta/\delta t$ and $u_{k,k}$. It is obvious from (26) that $\delta/\delta t$ is the time derivative as apparent to an observer moving with the velocity cX_i. Therefore $\delta/\delta t$ is known.

Equations (49) and (50) together with (47) give

$$(59)\quad H_{l,m} = K_{ij} M^i_l M^j_m$$

and

$$(60)\quad u_{l,m} = N_{ij} M^i_l M^j_m$$

In consequence of (59),(51),(52),(53) and (54),(2) gives

$$(61)\quad u_{k,k} = \Big[(G H_{n_1} + H^\beta \vartheta_{i\beta} X_i - u^\beta C_{i\beta} X_i) M^3_l M^3_l - C_{i\alpha}(M^\beta_l x^i_{,\beta} + M^3_l X_i) M^\alpha_l - F_\alpha M^\alpha_l M^3_l - \Big\{ F''_{i\alpha} M^\alpha_l M^3_l + (X_i - G''_i H_{n_1}) M^3_l M^3_l \Big\} \frac{\delta H_i}{\delta t} - \frac{H_{n_1} M^3_l M^3_l}{\rho(c-u_{n_1})} - (F'''_{i\alpha} M^\alpha_l M^3_l - H_{n_1} G'''_i M^3_l M^3_l) \frac{\delta u_i}{\delta t} \Big] \Big/ \Big\{ F'_\alpha M^\alpha_l M^3_l + H_{n_1}(1+G') M^3_l M^3_l \Big\}$$

Hence, quantities K_{ij} and N_{ij} are completely known.

The current density and the vorticity behind the shock are given by

$$(62)\quad J_i = \frac{1}{4\pi} \epsilon_{ijk} H_{k,j}$$

and

$$(63)\quad \omega_i = \epsilon_{ijk} u_{k,j}$$

which, in consequence of (59) and (60) give

$$(64)\quad J_i = \frac{1}{4\pi} \epsilon_{ijk} K_{lm} M^l_k M^m_j$$

and

$$(65)\quad \omega_i = \epsilon_{ijk} N_{lm} M^l_k M^m_j$$

5. GRADIENTS OF DENSITY AND PRESSURE.

By virtue of (24),(26) and (20),(3) gives

$$(66)\quad \rho_{,i} X_i = -\left(A_\alpha u^\alpha + \rho u_{k,k} + \frac{\delta \rho}{\delta t}\right) \Big/ (u_{n_1} - c)$$

Now, let us define quantities Z_i by

(67) $Z_i = \rho_{,\ell} L^{\ell}_{i}$

which,in consequence of (46),(20) and (66),gives

(68) $Z_{\alpha} = A_{\alpha}$

and

(69) $Z_3 = -\left(A_{\alpha} u^{\alpha} + \rho u_{k,k} + \frac{\delta \rho}{\delta t}\right) / (u_{n_j} - c)$

Hence,(67),with the help of (47),gives

(70) $\rho_{,\ell} = Z_i M^{i}_{\ell}$

Applying (26) to (4) and substituting for first derivatives of quantities from (59) and (60),we obtain the gradients of pressure.

6.CURVATURE OF STREAMLINES AND MAGNETIC LINES OF FORCE.

If the principal normal and tangent vectors to a curve be μ_i and t_i ,from Frenet's Formula,we have

(71) $K \mu_i = \frac{dt_i}{ds}$

where, s and K are the arc-length and curvature of the curve.Put- ting $t_i = \frac{H_i}{H}$, $\frac{d}{ds} = (H_j/H)(\partial/\partial x^j)$ and $t_i = \frac{u_i}{q}$ ↑, $(d/ds) = (u_j/u)($

and applying (59) and (60),we get

(72) $K \mu_i = \frac{K_{\ell m} M^{m}_{j} H_j}{H^2} \left\{ M^{\ell}_{i} - \frac{H_i H_k M^{\ell}_{k}}{H^2} \right\}$

and

(73) $K \mu_i = \frac{N_{\ell m} M^{m}_{j} u_j}{q^2} \left\{ M^{\ell}_{i} - \frac{u_i u_k M^{\ell}_{k}}{q^2} \right\}$

for the magnetic lines of force and the streamlines respectively.

7.SECOND PARTIAL DERIVATIVES OF u_i AND H_i .

In the preceding sections we have determined the first partial derivatives of u_i , H_i , p and ρ completely.The knowledge of the first derivatives can be applied to the determination of the second ones.

Let the derivatives of $H_{i,n}$, $u_{i,n}$, $p^{x}_{,i}$ and $\rho_{,i}$ with respect to y^{α} be given by

(74) $H_{i,nj} x^{j}_{,\alpha} = H_{i,jn} x^{j}_{,\alpha} = \psi_{in\alpha}$ (symmetric in j,n)

(75) $u_{i,nj} x^{j}_{,\alpha} = u_{i,jn} x^{j}_{,\alpha} = \psi'_{in\alpha}$ (symmetric in j,n)

↑ q is the magnitude of velocity-vector

(76) $p^{*}_{,ij}\, x^j_{,\alpha} = p^{*}_{,ji}\, x^j_{,\alpha} = \psi''_{i\alpha}$ (symmetric in i,j)

(77) $\rho_{,ij}\, x^j_{,\alpha} = \rho_{,ji}\, x^j_{,\alpha} = \psi'''_{i\alpha}$ (symmetric in i,j)

If we differentiate (1) with respect to x^n and apply (24),(25), (26),(74) and (75),we get

(78) $(c-u_{n/})H_{i,jn}X_j + H_{n/}u_{i,jn}X_j = \phi_{in} + H_i u_{k,kn} + \frac{\delta H_{i,n}}{\delta t}$

where,

(79) $\phi_{in} = u^{\beta}\psi_{in\beta} - H^{\beta}\psi'_{in\beta} + u_{j,n}H_{i,j} - H_{j,n}u_{i,j} + H_{i,n}u_{k,k}$

Multiplying (78) by $x^i_{,\alpha}$ and X_i, we get

(80) $(c-u_{n/})H_{i,jn}x^i_{,\alpha}X_j + H_{n/}u_{i,jn}X_j = \phi_{in}x^i_{,\alpha} + H_{\alpha}u_{k,kn} + \frac{\delta H_{i,n}}{\delta t}x^i_{,\alpha}$

and

(81) $(c-u_{n/})H_{i,jn}X_iX_j + H_{n/}u_{i,jn}X_iX_j = \phi_{in}X_i + H_{n/}u_{k,kn} + \frac{\delta H_{i,n}}{\delta t}X_i$

For evaluations of $H_{i,nj}x^i_{,\alpha}X_j$, $u_{i,nj}x^i_{,\alpha}X_j$ and $H_{i,nj}X_iX_j$, $u_{i,nj}X_iX_j$, we require two more equations containing these quantities.They can be obtained from (4) by eliminating $p^{*}_{,i}$ in two different ways.If we differentiate (4) with respect to x^n ,multiply it by $x^i_{,\alpha}$,substitute for $p^{*}_{,in}x^i_{,\alpha}$ from (76) and apply (24),(25),(26),(74) and (75),we get

(82) $\rho(c-u_{n/})u_{i,jn}x^i_{,\alpha}X_j + \frac{H_{n/}}{4\pi}H_{i,jn}x^i_{,\alpha}X_j = \xi_{n\alpha} + \frac{\delta u_{i,n}}{\delta t}x^i_{,\alpha}$

where,

(83) $\xi_{n\alpha} = \rho(u_{n/}-c)u^{\beta}\psi'_{in\beta}x^i_{,\alpha} - \frac{H^{\beta}}{4\pi}\psi_{in\beta}x^i_{,\alpha} - \frac{1}{4\pi}H_{j,n}H_{i,j}x^i_{,\alpha} + \rho u_{j,n}u_{i,j}x^i_{,\alpha}$

$$+\rho_{,n}(u_j - cX_j)u_{i,j}x^i_{,\alpha} + \rho_{,n}\frac{\delta u_i}{\delta t}x^i_{,\alpha} + \psi''_{n\alpha}$$

Solving (80) and (82) for $u_{i,jn}x^i_{,\alpha}X_j$ and $H_{i,jn}x^i_{,\alpha}X_j$,we get

(84) $H_{i,jn}x^i_{,\alpha}X_j = U_{n\alpha} + U'_{i\alpha}\frac{\delta H_{i,n}}{\delta t} + U''_{i\alpha}\frac{\delta u_{i,n}}{\delta t} + U'''_{\alpha}u_{k,kn}$

and

(85) $u_{i,jn}x^i_{,\alpha}X_j = V_{n\alpha} + V'_{i\alpha}\frac{\delta H_{i,n}}{\delta t} + V''_{i\alpha}\frac{\delta u_{i,n}}{\delta t} + V'''_{\alpha}u_{k,kn}$

where,

(86) $U_{n\alpha} = \frac{4\pi\{H_{n/}\xi_{n\alpha} - \rho(c-u_{n/})\phi_{in}x^i_{,\alpha}\}}{H^2_{n/} - 4\pi\rho(c-u_{n/})^2}$

(87) $U'_{i\alpha} = \frac{4\pi(u_{n/}-c)x^i_{,\alpha}}{H^2_{n/} - 4\pi\rho(c-u_{n/})^2}$

(88) $$U''_{i\alpha} = \frac{4\pi\rho H_{\eta} x^i_{,\alpha}}{H_\eta^2 - 4\pi\rho(c-u_\eta)^2}$$

(89) $$U'''_\alpha = \frac{4\pi\rho(u_\eta - c)H_\alpha}{H_\eta^2-4\pi\rho(c-u_\eta)^2}$$

and

(90) $$V_{n\alpha} = \frac{H_\eta \varphi_{in} x^i_{,\alpha} - 4\pi(c-u_\eta)\xi_{n\alpha}}{H_\eta^2 - 4\pi\rho(c-u_\eta)^2}$$

(91) $$V'_{i\alpha} = \frac{H_\eta x^i_{,\alpha}}{H_\eta^2 - 4\pi\rho(c-u_\eta)^2}$$

(92) $$V''_{i\alpha} = \frac{4\pi\rho(u_\eta - c)x^i_{,\alpha}}{H_\eta^2 - 4\pi\rho(c-u_\eta)^2}$$

(93) $$V'''_\alpha = \frac{H_\eta H_\alpha}{H_\eta^2 - 4\pi\rho(c-u_\eta)^2}$$

Differentiating (6) materially and applying (3) and (5) we get

(94) $$\frac{\partial p}{\partial t} + u_i p_{,i} = -\gamma p u_{k,k}$$

Again, differentiating it partially with respect to x^n and applying (26), we get

(95) $$(cX_i - u_i)p_{,in} = \frac{\delta p_{,n}}{\delta t} + \gamma p u_{k,kn}$$

Now, differentiating (4) partially with respect to x^n, multiplying it by $cX_i - u_i$, substituting for $(cX_i-u_i)p_{,in}$ from (95) and applying (26), (24), (25), (74), (75), (84) and (85), we get

(96) $$u_{i,jn}X_iX_j = W_n + W'_i \frac{\delta H_{i,n}}{\delta t} + W''_i\frac{\delta u_{i,n}}{\delta t} + W''' u_{k,kn} - \frac{1}{\rho(c-u_\eta)^2}\frac{\delta p_{,n}}{\delta t}$$

where,

(97) $$\begin{aligned} W_n = \Big[& 4\pi\rho(c-u_\eta)u^\alpha V_{n\alpha} + \{H_\eta u^\alpha + H^\alpha(u_\eta - c)\}U_{n\alpha} + (cX_i + H_i - u_i)\varphi_{in\alpha}u^\alpha \\ & + 4\pi\rho(cX_i - u_i)u^\alpha \psi^i_{n\alpha} + 4\pi\rho\,\rho_{,n}(cX_i-u_i)(cX_i-u_j)u_{i,j} \\ & - 4\pi\rho(cX_i-u_i)u_{j,n}u_{i,j} + (cX_i-u_i)H_{j,n}H_{i,j} + (u_i - cX_i)H_{j,n}H_{j,i} \\ & - 4\pi\gamma p_{,n}u_{k,k} - 4\pi p_{,i}u_{i,n} - 4\pi\rho_{,n}(cX_i-u_i)\frac{\delta u_i}{\delta t}\Big] \Big/ 4\pi\rho(c-u_\eta)^2 \end{aligned}$$

(98) $W'_i = 4\pi\rho(c-u_{n'})V'_{i\alpha}u^\alpha + \{H_{n'}u^\alpha + H^\alpha(u_{n'}-c)\}U'_{i\alpha} \Big/ 4\pi\rho(c-u_{n'})^2$

(99) $$W''_i = \frac{4\pi\rho(c-u_{n'})u^\alpha V''_{i\alpha} + \{H_{n'}H^\alpha + H^\alpha(u_{n'}-c)\}U''_{i\alpha} - 4\pi\rho(cX_i - u_i)}{4\pi\rho(c-u_{n'})^2}$$

(100) $$W''' = \frac{4\pi\rho(c-u_{n'})u^\alpha V'''_\alpha + \{H_{n'}u^\alpha + H^\alpha(u_{n'}-c)\}U'''_\alpha - 4\pi\gamma p}{4\pi\rho(c-u_{n'})^2}$$

In consequence of (96),(81) gives

(101) $$H_{i,jn}X_iX_j = \frac{1}{(c-u_{n'})}\Big\{\varphi_{in}X_i - H_{n'}W_n + (X_i - H_{n'}W'_i)\frac{\delta H_{i,n}}{\delta t} + \frac{H_{n'}}{\rho(c-u_{n'})^2}\frac{\delta p_{,n}}{\delta t} - H_{n'}W''_i\frac{\delta u_{i,n}}{\delta t} + H_{n'}(1-W''')u_{k,kn}\Big\}$$

Now let us define quantities K'_{ijk} and N'_{ijk} by

(102) $K'_{ijk} = H_{l,mn}L^l_i L^m_j L^n_k$ (symmetric in j,k)

and

(103) $N'_{ijk} = u_{l,mn}L^l_i L^m_j L^n_k$. (symmetric in j,k)

With the help of (46),(74),(75),(85),(96) and(97),(102) and (103) give

(104) $K'_{ij\alpha} = \psi_{lm\alpha}L^l_i L^m_j$

(105) $K'_{\alpha 33} = U_{n\alpha}X_n + U'_{i\alpha}\frac{\delta H_{i,n}}{\delta t}X_n + U''_{i\alpha}\frac{\delta u_{i,n}}{\delta t}X_n + U'''_\alpha u_{k,kn}X_n$

(106) $$K'_{333} = \frac{X_n}{c-u_{n'}}\Big\{\varphi_{in}X_i - H_{n'}W_n + (X_i - H_{n'}W'_i)\frac{\delta H_{i,n}}{\delta t} - H_{n'}W''_i\frac{\delta u_{i,n}}{\delta t}$$

and

(107) $N'_{ij\alpha} = \psi'_{lm\alpha}L^l_i L^m_j$

(108) $N'_{\alpha 33} = V_{n\alpha}X_n + V'_{i\alpha}\frac{\delta H_{i,n}}{\delta t}X_n + V''_{i\alpha}\frac{\delta u_{i,n}}{\delta t}X_n + V'''_\alpha u_{k,kn}X_n$

(109) $N'_{333} = W_nX_n + W'_i\frac{\delta H_{i,n}}{\delta t}X_n + W''_i\frac{\delta u_{i,n}}{\delta t}X_n + W'''u_{k,kn}X_n - \frac{1}{\rho(c-u_{n'})^2}\frac{\delta p_{,n}}{\delta t}X_n$

K'_{ijk} and N'_{ijk} are known if $K'_{ij\alpha}$, K'_{ij3} and $N'_{ij\alpha}$, N'_{ij3}, are known. Because of symmetry in the last two indices, K'_{ij3} and N'_{ij3} are equivalent to K'_{i3j} and N'_{i3j}. Therefore K'_{ij3} and N'_{ij3} are known if $K'_{i3\alpha}$, K'_{i33} and $N'_{i3\alpha}$, N'_{i33} are known. Again, the determinations of K'_{i33} and N'_{i33} are equivalent to the determinations of $K'_{\alpha 33}$, K'_{333} and $N'_{\alpha 33}$, N'_{333}. Hence, K'_{ijk} and N'_{ijk} can be determined by determining $K'_{ij\alpha}$, $K'_{i3\alpha}$, K' and $N'_{ij\alpha}$, $N'_{i3\alpha}$, N'_{333}, which are given by (104), (105), (106), (107), (108) and (109).

In consequence of (47), (102) and (103) give

$$(110)\quad H_{lmn} = K'_{ijk} m^i_l m^j_m m^k_n$$

and

$$(111)\quad u_{l,mn} = N'_{ijk} m^i_l m^j_m m^k_n$$

Now, differentiating (2) partially with respect to x^n, multiplying by X_n, substituting from (110) and applying (104), (105) and (106), we get an equation containing $u_{k,kn}X_n$. The quantity $u_{k,kn}X_n$ can be evaluated from this equation. (110) and (111) give the second partial derivatives of the velocity and the magnetic field vectors in terms of the known quantities.

8. SECOND PARTIAL DERIVATIVES OF DENSITY AND PRESSURE.

For the determination of the second derivative of the density, the proper form of the equation of continuity is required. Differentiating (3) partially with respect to x^n and applying (26), (24) and (77), we get

$$(112)\quad \rho_{,in} X_i = R_n + \frac{\rho}{(c-u_n)} u_{k,kn} + \frac{1}{(c-u_n)} \frac{\partial \rho_{,n}}{\partial t}$$

where,

$$(113)\quad R_n = \frac{u_i^\alpha \psi'''_{n\alpha} + u_{i,n}\rho_{,i} + \rho_{,n} u_{k,k}}{(c-u_n)}$$

Now, let us define quantities Z'_{ij} by

$$(114)\quad Z'_{ij} = \rho_{,lm} L^l_i L^m_j \qquad \text{(symmetric in i, j)}$$

which, in consequence of (46), (77) and (112), gives

$$(115)\quad Z'_{i\alpha} = \psi'''_{l\alpha} L^l_i$$

and

$$(116)\quad Z'_{i3} = \left\{ R_l + \frac{\rho}{(c-u_n)} u_{k,kl} + \frac{1}{c-u_n} \frac{\partial \rho_{,l}}{\partial t} \right\} L^l_i$$

(115) and (116) determine quantities Z'_{ij} completely. Hence, by virtue

(47),(114) gives

(117) $p_{,lm} = Z'_{ij} M^i_l M^j_m$

To determine the second partial derivative of pressure, we shall take help of (4). If we differentiate it partially with respect to x^n, make use of (26) and substitute for the second partial derivatives of the velocity and the magnetic field vectors, the second derivative of the pressure is obtained.

9. TORSION OF STREAMLINES AND MAGNETIC LINES OF FORCE.

With the help of the notations used in section 6, the torsion of the streamlines is given by

$$(118) \qquad \tau = -\frac{1}{K^2}\begin{vmatrix} \frac{u_1}{q} & \frac{u_2}{q} & \frac{u_3}{q} \\ \frac{\partial u_1}{\partial s} & \frac{\partial u_2}{\partial s} & \frac{\partial u_3}{\partial s} \\ \frac{\partial^2 u_1}{\partial s^2} & \frac{\partial^2 u_2}{\partial s^2} & \frac{\partial^2 u_3}{\partial s^2} \end{vmatrix} = -\frac{1}{K^2}\epsilon_{ijk}\frac{u_i}{q}\frac{\partial u_j}{\partial s}\frac{\partial^2 u_k}{\partial s^2}$$

But

$$(119) \qquad \frac{\partial u_i}{\partial s} = \frac{1}{q} u_j u_{i,j}$$

and

$$(120) \qquad \frac{\partial^2 u_i}{\partial s^2} = \frac{u_k}{q}\frac{\partial}{\partial x^k}\left(\frac{u_j u_{i,j}}{q}\right) = \frac{u_n}{q^4}\left\{q^2 u_r u_{i,rn} + q^2 u_{i,r} u_{r,n} - u_r u_l u_{i,r} u_{l,n}\right\}$$

Therefore, with the help of (118), (119) and (120), we get

$$(121) \qquad \tau = -\frac{1}{q^6 K^2}\epsilon_{ijk} u_i u_m u_{j,m} u_s \left\{q^2 u_r u_{k,rs} + q^2 u_{k,r} u_{r,s} - u_r u_l u_{k,r} u_{l,s}\right\}$$

All the terms of the right hand side, being in terms of the first and the second derivatives, are known. Similarly, with the help of the knowledge of the derivatives of the magnetic field vector, the torsion of a magnetic line of force can be obtained from

$$(122) \qquad \tau = \frac{-1}{H^6 K^2}\epsilon_{ijk} H_i H_m H_{j,m} H_s \left\{H^2 H_r H_{k,rs} + H^2 H_{k,r} H_{r,s} - H_r H_l H_{k,r} H_{l,s}\right\}$$

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